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THE STOCHASTIC FILTERING PROBLEM FOR POINT PROCESSES

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The Stochastic Filtering Problem for Point Processes*)

by

J.H. van Schuppen

ABSTRACT

The purpose of this paper is to give a brief exposition of the stochastic filtering problem in the case of point process observations. Some examples of these problems will be presented. We discuss the modelling of point processes and formulate the associated stochastic dynamical systems. Two methods to resolve the stochastic filtering problem will be given, namely the semimartingale representation method and the measure transformation method. For several examples the solution to the stochastic filtering problem will be given. Some open questions are mentioned.

KEY WORDS & PHRASES: *Point processes, Marked point processes, stochastic analysis, stochastic dynamical systems, stochastic filtering problems.*

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1. INTRODUCTION

The purpose of this paper is to give a brief exposition of the stochastic filtering problem for point process observations. No proofs and few details will be presented. However, references to the literature have been provided.

For many practical filtering problems the observations are not continuous processes, but point processes or marked point processes. Examples of such problems arise in the areas of optical communication, nuclear medicine, urban traffic control, and operations research. Many questions in these areas may be formulated as stochastic filtering problems or stochastic control problems.

What results have been obtained for these problems? Representations for point processes have been obtained using concepts from the theory of stochastic processes. These representations may be considered to be stochastic dynamical systems. For the stochastic filtering problem for these systems two methods have been developed. Both of these methods yield general representation results that have to be applied to specific models. For several models filtering algorithms have been obtained. A brief summary of these results is presented below.

For the material on point processes we refer to the books [16, 18, 24], and for application oriented books we suggest [21, 35].

2. MOTIVATING EXAMPLES

Some examples of stochastic filtering problems with point process observations are presented below.

EXAMPLE 2.1. Optical Communication

The physical model is as follows. A signal modulates an optical source, which in turn generates an optical beam. The beam, after travelling a certain distance, is incident on a detector. The optimal detector produces an electric current which we regard as the observed process.

The problem then is to estimate the signal on the basis of the observed process.

Particular cases of the above problem occur in:

1) experiments in nuclear physics;
2) light scattering studies;
3) optical communication with lasers.

For references on these problems see [19, 20, 22, 33].

EXAMPLE 2.2. An Industrial Problem

The model is that of a machine that irregularly produces unit products. Initially the machine operates at full capacity. At some point in time the machine breaks down partially, with the effect that it yields output at a lower rate. One may then associate a cost function with this model, such that under normal conditions a profit is made and a loss when the machine is partially defective.

Then problem is then:

1) to estimate when the machine breaks down partially on the basis of the production data only;
2) to resolve the stochastic control problem of when to shut down the machine so as to minimize the costs.

The first problem mentioned above is also known as the Poisson disorder problem. For references see [12, 40].

EXAMPLE 2.3. Traffic Estimation

The model is that of an urban traffic network. Information on the traffic flow is obtained from detection lines. The ultimate objective is computer control of urban traffic.

The problem here is to estimate and to predict traffic intensities.

For some references on this problem see [1, 2, 36].

EXAMPLE 2.4. The Firefly Model

The model is for a swarm of flireflies. Each of the fireflies irregularly produces flashes of light. One assumes that the swarm has a Gaussian density, and that the mean of this density moves around according to a Gauss-Markov process. The authors of this model leave it to the imagination of the reader to think of other applications of this model.

The problem is to estimate the mean of the density on the basis of the observations of the light flashes [41].

EXAMPLE 2.5. Filtering in Queueing Problems

Consider a model for a waiting line, in which we distinguish an arrival and a departure process, and the queue process. One may also consider a network of queues.

The problem is to estimate the queue process on the basis of the departure process. The theory of stochastic filtering is also used to prove a certain result for so-called Jackson networks.

For references on the application of stochastic filtering to queueing problems see [7, 9].

EXAMPLE 2.6. A Miscellaneous Model

We finally mention a model that has been addressed in the literature. Here the signal process x is a finite state or denumerable state Markov process. The observed process y is denumerably valued and related to x by the equation $y_t = h(t,x_t)$.

The problem then is to estimate the signal process given past observations. For references on this problem see [25, 26]. In this paper we will not discuss this model any further.

3. MODELLING OF JUMP PROCESSES

In this section we answer the question: How to model a jump process? In the sequel we limit attention to continuous time processes. We assume given a complete probability space (Ω,F,P). In this paper we will not be very detailed about technical conditions. The reader is referred to the references [3, 10, 13, 18, 21, 23] for definitions and results.

3.1. Single Unit Jump Case

We start the modelling of jump processes with a rather elementary case. Let $x\colon \Omega \times T \to R$, on $T = R_+$, be a process with a single unit jump, with $x_0 = 0$. Define the jump time as

$$\tau(\omega) := \begin{cases} \inf\{t \in T \mid x_t(\omega) \neq x_0(\omega)\}, \\ +\infty \qquad \text{if } x_t(\omega) = x_0(\omega) \text{ for all } \omega \in \Omega, \end{cases}$$

and the jump distribution function as $f(t) = P(\{\tau \le t\})$.

We then have the following characterization.

THEOREM 3.1.

(a) *Given the single unit jump process described above. Then there exists an unique process* $\bar{x}\colon \Omega \times T \to \overline{R_+}$ *such that*

$$x = \bar{x} + m \tag{*}$$

where $(m_t, F_t, t \in T) \in M_1$, *meaning it is a martingale. Actually* $\bar{x}$ *is given by*

$$\bar{x}_t = \int_{(0,t\wedge\tau]} [1 - f(u-)]^{-1} f(du).$$

Here $(F_t, t \in T)$ *is an increasing* σ*-algebra family.*

(b) *The process* $\bar{x}$ *uniquely characterizes the measure* P. [11, 18:III].

The decomposition (*) above is called the special semimartingale decomposition. This decomposition will play a fundamental role throughout this discussion.

To further clarify jump processes with a single unit jump, we illustrate the classification of jump times, as introduced in [13], with some examples.

1. Totally inaccessible jump times. Example: $f(t) = 1 - \exp(-t)$. Here the distribution of the jump is diffuse on R_+.
2. Predictable jump times. Example: Jump distribution $f(t) = I_{[c,\infty)}(t)$, for some $c \in R_+$. Intuitively this jump can be predicted with certainty.

3. Accessible jump times. Example: Jump distribution $f(t) = \Sigma_{k=1}^{n} \alpha_k I_{[u_k,\infty)}(t)$ with $0 < u_1 < u_2 < \ldots < u_n < \infty$, and $\alpha_k \in R$ with $\Sigma_{k=1}^{n} \alpha_k = 1$. In this case the jump is not predictable but can only occur at certain time moments.

From the above classification one may deduce that in general one uses totally inaccessible jump times.

3.2. The Counting Process Case

The modelling of jump processes introduced above may be extended to counting processes. A stochastic process is a counting process if it starts at time zero, is piecewise constant, and has positive unit jumps. By convention it is taken to be right continuous. A counting process is also called a point process.

THEOREM 3.2.1. *Given a counting process* $(n_t, F_t, t \in T)$.

(a) *There exists an unique increasing predictable process* $a: \Omega \times T \to R_+$ *such that*

$$n = a + m \tag{*}$$

with $(m_t, F_t, t \in T) \in M_{1uloc}$, *meaning* m *is a local martingale* [23].

(b) *If there exists a process* $\lambda: \Omega \times F \to R_+$ *such that*

$$a_t = \int_0^t \lambda_s ds$$

then we call λ *the rate process associated with* $(n_t, F_t, t \in T)$. *In this case we obtain the representation*

$$dn_t = \lambda_t dt + dm_t, \qquad n_0 = 0. \tag{**}$$

(c) *The process* a *characterizes uniquely the measure* P *with respect to the counting process* n.

References [11, 18]. The decomposition (*) and (**) above are called special semimartingale decompositions.

EXAMPLE 3.2.2. Let $n: \Omega \times T \to R$, $\lambda: \Omega \times T \to R_+$ be processes such that

$$E[\exp(iv(n_t - n_s)) \mid F_s^n \vee F_\infty^\lambda] = \exp((e^{iv}-1)(\int_s^t \lambda_\tau d\tau)).$$

Thus, conditioned on F_∞^λ, n is a Poisson process with rate λ. Such a process is called a *doubly stochastic Poisson process*. This description is equivalent with the description

$$dn_t = \lambda_t dt + dm_t, \qquad n_0 = 0,$$

with $(m_t, F_t^n \vee F_\infty^\lambda, t \in T) \in M_{1uloc}$.

3.3. Arbitrary Jump Processes

We first mention several descriptions of jump processes.

1. The *marked point process* description: given $\{x_n, s_n, n \in Z_+\}$, where s_n represents the interarrival time between the n-1 and the n-th jump, and x_n the value or mark at the n-th jump. Let $\tau_0 = 0$, and for $n \in Z_+$ $\tau_n = \Sigma_{k=1}^n s_k$, to be called the n-th jump time.
2. The *jump process* description: given the process $x: \Omega \times T \to R$ such that

$$x_t = \sum_{n \in Z_+} x_n I_{(\tau_n \leq t < \tau_{n-1})}.$$

3. The *jump measure* description: given the random measure $p: \Omega \times B_T \otimes B \to \overline{R_+}$,

$$p(\omega, A) = \sum_{n \in Z_+} I_A(\tau_n(\omega), x_n(\omega)).$$

The above descriptions can be shown to be equivalent. The sought for characterization reads then as follows.

THEOREM 3.3.1. *Given a jump measure* $p: \Omega \times B_T \otimes B \to \overline{R_+}$.

(a) *There exists an unique predictable random measure* $\bar{p}: \Omega \times B_T \otimes B \to \overline{R_+}$ *such that*

$$p(\omega, dt \times dz) = \bar{p}(\omega, dt \times dz) + q(\omega, dt \times dz)$$

with $(q(\omega, 0,t] \times A), F_t, t \in T) \in M_{1uloc}$ *for all* $A \in B$.
(b) *The random measure* $\bar{p}$ *uniquely characterizes the measure* P.
[18:III].

3.4. Modelling of the Rate Process

Consider a counting process n which admits a rate process λ, say with representation

$$dn_t = \lambda_t dt + dm_t, \qquad n_0 = 0.$$

The question then is how to model the rate process. We present several models for the rate process that are used in the literature.

1. The constant rate process: $\lambda_t = \lambda_0$ for all $t \in T$, with some distribution for λ_0 specified.
2. The rate process as a finite or denumerable state Markov process.
3. The energy model: $\lambda_t = \mu_0 + \mu_1 x_t^2$ with $\mu_0, \mu_1 \in (0,\infty)$ and x a Gauss-Markov or a diffusion process.
4. The linear model: $\lambda_t = \mu_0 + \mu_1 x_t$, with $\mu_0, \mu_1 \in (0,\infty)$, and x a diffusion process. The problem with this model is that the rate process has to be stopped if it becomes negative; hence it is not a useful model.

As an illustration we present one example of a model for a counting process.

EXAMPLE 3.4.1. The Poisson-FSMP Model. Given the process, $n: \Omega \times T \to R$, with $n_0 = 0$,

$$E[\exp(iv(n_t - n_s)) \mid F_s^n \vee F_\infty^\lambda] = \exp((e^{iv}-1)(\int_s^t \lambda_\tau d\tau)),$$

for all $t > s$, $v \in R$,

and the process $\lambda: \Omega \times T \to X := \{r_1, r_2, \ldots, r_m\} \subset (0,\infty)$ which is a finite state Markov process on $(\lambda_t, F_t^n \vee F_t^\lambda, t \in T)$. Under certain differentiability conditions on the semigroup of the Markov process we may represent these processes as

$$dz_t = A(t)z_t dt + \phi(t,0)dm_{1t}, z_0,$$

$$dn_t = D\, z_t dt + dm_t, \qquad n_o = 0,$$

$$\lambda_t = D\, z_t,$$

where

$$z: \Omega \times T \to R^m, \quad z_t^i := I_{(\lambda_t = r_i)},$$

$$D := (r_1 r_2 \ldots r_m) \in R^{1\times m}, \quad \phi: T \times T \to R^{m\times m},$$

$$\phi_{ij}(t,s) := \begin{cases} E[z_t^i z_s^j]/E[z_s^j], & \text{if } E[z_s^i] > 0,\ s \le t, \\ 0, & \text{otherwise}, \end{cases}$$

$$A(t) := \lim_{s\to t} [\phi(t,s) - I]/(t-s),$$

$$(m_t, F_t^n \vee F_t^\lambda,\ t \in T) \in M_{1uloc}, \quad (m_{1t}, F_t^n \vee F_t^\lambda,\ t \in T) \in M_1.$$

3.5. Stochastic Dynamical Systems

The preceding representations of jump processes may be considered as stochastic dynamical systems. The concept of a stochastic dynamical system we define below. For a discussion of this notion and a formulation of the stochastic realization problem see [37].

DEFINITION 3.5.1. A *stochastic dynamical system* (in continuous time) is a collection

$$\{\Omega, F, P, T, Y, B_y, \underline{Y}, X, B_x\}$$

where $\{\Omega,F,P\}$ is a complete probability space, $T \subset R$ an interval, $\{Y,B_y\}$ a measurable space with Y a vector space, $\underline{Y} \subset \{y: T \to Y\}$, $\{X,B_x\}$ a measurable space, such that if $x: \Omega \times T \to X$, $y: \Omega \times T \to Y$ are stochastic processes then

$$({}_tF^{\Delta y} \vee {}_tF^x,\ F^{x_t}, F_t^x \vee F_t^y) \in CI$$

for all $t \in T$; equivalently, if

$$E[I_{A_1} I_{A_2} \mid F_t^x \vee F_t^y] = E[I_{A_1} I_{A_2} \mid F^{x_t}]$$

for all $A_1 \in {}_tF^{\Delta y}$, $A_2 \in {}_tF^x$, $t \in T$. Here

$$F^{x_t} = \sigma(\{x_t\}), \quad F_t^x = \bigvee_{s \le t} F^{x_s}, \quad {}_tF^x = \bigvee_{s > t} F^{x_s},$$

$${}_tF^{\Delta y} = \sigma(\{y_s - y_t, \forall s > t\}),$$

$$F_t^y = \bigvee_{s \le t} F^{y_s}.$$

NOTATIONS. $(\Omega,F,P,T,Y,B_y,\underline{Y},X,B_x) \in \Sigma S$, and we call x the *state process* and y the *output process*.

In the sequel we assume that $Y = R^k$, $B_y = B_k$ the Borel σ-algebra on Y, $X = R^n$, and $B_x = B_n$.

The above definition expresses that the distribution of a future state and a future output increment conditioned on past states and past outputs, depends only on the current state. This property is the characteristic of a stochastic dynamical systems. An immediate consequence of this definition is that the state process is a Markov process. For a stochastic dynamical system one may also formulate the concepts of stochastic observability and stochastic reconstructability, see [37]. However for a number of concepts related to a stochastic dynamical system precise formulations are not yet clear.

4. THE STOCHASTIC FILTERING PROBLEM

DEFINITION 4.1. Given a stochastic dynamical system,

$$(\Omega,F,P,T,Y,B_y,\underline{Y},X,B_x) \in \Sigma S.$$

(a) The *stochastic filtering problem* for this system is to determine

$$E[\exp(iu'x_t) \mid F_t^y]$$

for all $u \in R^n$, $t \in T$.

(b) A *past output based filter system* for the stochastic filtering problem defined above is a stochastic dynamical system

$$(\Omega, F, P, T, Y, B_y, \underline{Y}, Z, B_z) \in \Sigma S$$

such that if $y\colon \Omega \times T \to Y$, $z\colon \Omega \times T \to Z$ are the underlying processes then $F^{z_t} \subset F_t^y$ for all $t \in T$.

In part (a) above to determine the conditional characteristic function means to exhibit the analytic form of the map

$$y_{[0,t]} \to E[\exp(iu'x_t) \mid F_t^y].$$

A filter system always exists, because we can take $Z = \underline{Y}$. It is therefore of interest to find a filter state space Z which is in some sense minimal. The concept of the dimension of such a state space is not yet clear. For a stochastic filtering problem with continuous observations it has been suggested to relate the dimension of a filter system to the dimension of the Lie algebra associated with the operators occurring in the equation for the conditional density. For this issue there still are many open questions.

How to resolve the stochastic filtering problem? The general procedure is to derive an equation for $E[\exp(iu'x_t) \mid F_t^y]$, and to solve this equation. We present two methods to effect this program.

5. THE SEMIMARTINGALE REPRESENTATION METHOD

In this section we present the semimartingale representation method to resolve the stochastic filtering problem. Initially we do not work with the state process but with an arbitrary special semimartingale process. The problem we then consider is to find the special semimartingale decomposition of the projection of this process on the σ-algebra family generated by the observations. For specific stochastic dynamical systems this abstract representation can be applied to yield a partial stochastic differential equation for the conditional characteristic function. Below we first state two abstract representation results, and subsequently show how these results are applied to obtain filtering algorithms.

5.1. The Counting Process Case

We summarize the main result. For a precise statement see the references mentioned below.

MODEL 5.1.1. Given a counting process model

$$dx_t = f_t dt + dm_{1t}, \qquad x_0,$$

$$dn_t = \lambda_t dt + dm_t, \qquad n_0 = 0,$$

$$d\langle m_1, m\rangle_t = \phi_t dt,$$

where n represents a counting process, assumed to have totally inaccessible jump times, λ the rate process, and x is a semimartingale with the indicated decomposition.

PROBLEM 5.1.2. To obtain the special semimartingale representation of the projection of the process x on the σ-algebra family $(F_t^n, t \in T)$. For the projection we take the so-called optional projection [13], which we denote by $(\hat{x}_t, t \in T)$. Then it follows that $\hat{x}_t = E[x_t \mid F_t^n]$ a.s. for all $t \in T$.

RESULT 5.1.3. The solution to the above formulated problem is given by

$$d\hat{x}_t = \hat{f}_t dt + [\hat{\Sigma}_{t-}^{x\lambda} + \hat{\phi}_{t-}]\hat{\lambda}_{t-}^{-1}(dn_t - \hat{\lambda}_t dt), \qquad \hat{x}_0 = E(x_0)$$

$$\hat{\Sigma}_t^{x\lambda} = E[(x_t - \hat{x}_t)(\lambda_t - \hat{\lambda}_t) \mid F_t^n].$$

References [6, 21, 27, 29, 30, 31, 38, 39].

5.2. The Jump Process Case

MODEL 5.2.1. Given the processes

$$x = x_0 + a + m$$

$$p(\omega, dt\times dz) = h(\omega,t,z)\mu(\omega,dt\times dz) + q(\omega,dt\times dz),$$

$$d\langle m, q(\omega,(0,t]\times A)\rangle_t = \int_A \psi(\omega,s,z)\mu(\omega,dt\times dz),$$

where x is a semimartingale, the second line a jump measure description of a jump process, and the third line represents the relation between the jump process and the semimartingale. We assume that the jump times are totally inaccessible.

PROBLEM 5.2.2. To determine the semimartingale representation of the projection of the process x on the σ-algebra family generated by the jump process.

RESULT 5.2.3. The solution to the above formulated problem is given by the representation

$$\hat{x}_t = \hat{x}_0 + \bar{a}_t + \int_0^t \int_R \hat{k}(\omega,s-,z)\bar{q}(\omega,ds\times dz),$$

$$\hat{q}(\omega,dt\times dz) = p(\omega,dt\times dz) - \hat{h}(\omega,s,z)\mu(\omega,dt\times dz),$$

$$\hat{k}(\omega,t,z) = [E[(x_t-\hat{x}_t)(h(\omega,t,z) - \hat{h}(\omega,t,z)) \mid F_t^y]$$

$$+ E[\psi(\omega,t,z)h(\omega,t,z) \mid F_t^y]]\hat{h}^{-1}(\omega,t,z).$$

For a precise statement of this result see [3]. Related references are [8, 17, 41].

5.3. Examples

We present the solutions to stochastic filtering problems for certain examples.

EXAMPLE 5.3.1. The Poisson-FSMP model. This model has been formulated in 3.4.1 and reads as follows,

$$dz_t = A(t)z_t dt + \phi(t,0)dm_{1t}, \qquad z_0,$$

$$dn_t = D\, z_t dt + dm_t, \qquad n_0 = 0.$$

$$\lambda_t = D\, z_t.$$

The solution to the stochastic filtering problem for this model is

$$E[\exp(iu\lambda_t) \mid F_t^n] = \sum_{j=1}^{m} \exp(iu_j r_j)\hat{z}_t^j,$$

$$d\hat{z}_t = A(t)\hat{z}_t dt + K(\hat{z}_{t-})(D\hat{z}_{t-})^{-1}(dn_t - D\hat{z}_t dt), \qquad \hat{z}_0 = E(z_0),$$

$$K(\hat{z}_t) = [\text{diagonal}(\hat{z}_t) - \hat{z}_t\hat{z}_t']D',$$

$$\hat{\lambda}_t = D\hat{z}_t.$$

Reference [28]. The above result can be extended to the case where λ is a denumerable state Markov process. An application of this model is in the estimation problem for the industrial model formulated in (2.2).

EXAMPLE 5.3.2. The Poisson-Gamma model. This model is rather elementary. We use it to illustrate the solution procedure for the stochastic filtering problem by the semimartingale representation method. The model is specified by

$$dx_t = \alpha x_t dt, \qquad x_0,$$

$$dn_t = x_t dt + dm_t, \qquad n_0 = 0,$$

where n is a counting process, x its associated rate process, $\alpha \in (-\infty,0)$, $x_0\colon \Omega \to R_+$ a random variable with a Gamma distribution function, with density function $p(v) = \beta^{-r}v^{r+1}e^{-v/\beta}/\Gamma(r)$, $r,\ \beta \in (0,\infty)$. Of course $x_t = \exp(\alpha t)x_0$.

We sketch the solution procedure. Set

$$c:\ \Omega \times T \times R \to C,$$

$$c_t(u) := \exp(iux_t).$$

Then

$$dc_t(u) = iu\alpha x_t c_t(u)dt, \qquad c_0(u).$$

Applying the semimartingale representation result to the process $c(u)$ we obtain

$$d\hat{c}_t(u) = u\alpha\hat{c}'_t(u)dt + [-i\hat{c}'_t(u) + i\hat{c}'_{t-}(0)\hat{c}_{t-}(u)](-i\hat{c}'_{t-}(0))^{-1}$$

$$(dn_t + i\hat{c}'_{t-}(0)dt),$$

$$\hat{c}_0(u) = E[\exp(iux_0)] = (1 - iu\beta)^{-r},$$

$$\hat{c}'_t(u) = \partial\hat{c}_t(u)/\partial u.$$

This is a partial stochastic differential equation, driven by a counting process.

The solution to the stochastic filtering problem for the above model then is

$$E[\exp(iux_t) \mid F^n_t] = (1 - iu\beta(t))^{-(n_t+r)},$$

$$\dot{\beta}(t) = \alpha\beta(t) - \beta^2(t), \qquad \beta(0) = \beta.$$

Then it follows that

$$\hat{x}_t = E[x_t \mid F^n_t] = \beta(t)(n_t + r),$$

satisfies the stochastic differential equation,

$$d\hat{x}_t = \alpha\hat{x}_t dt + \beta(t)(dn_t - \hat{x}_t dt), \qquad \hat{x}_0 = r\beta.$$

References [15, 27].

<u>EXAMPLE 5.3.3</u>. The firefly model: according to (2.4) we have

$$dx_t = \alpha x_t dt + \beta dv_t, \qquad x_0,$$

$$p(\omega, dt\times dz) = (2\pi\sigma(t)^2)^{-\frac{1}{2}}\exp(-(z-x_t)^2/2\sigma(t)^2)dtdz + q(\omega, dt\times dz),$$

where $\alpha \in R_-$, $\beta \in R$, v is a standard Brownian motion process, $x_0 : \Omega \to R$ is a random variable with a Gaussian distribution. The solution to the stochastic filtering problem is

$$E[\exp(iux_t) \mid F_t^y] = \exp(iu\hat{x}_t - \frac{1}{2}u^2\hat{r}_t),$$

$$d\hat{x}_t = \alpha\hat{x}_t dt + k_{t-}\int_R (z-\hat{x}_{t-})\bar{q}(\omega, dt\times dz),$$

$$d\hat{r}_t = [2\alpha\hat{r}_t + \beta^2 - \hat{r}_t^2(\sigma(t)^2 + \hat{r}_t)^{-1}]dt + k_{t-}\hat{r}_{t-}\int_R \bar{q}(\omega, dt\times dz),$$

$$\bar{q}(\omega, dt\times dz) = p(\omega, dt\times dz) - (2\pi(\sigma(t)^2+\hat{r}_t))^{-\frac{1}{2}}\exp(-(z-\hat{x}_t)^2/$$

$$/(2\pi(\sigma(t)^2 + \hat{r}_t)))dtdz,$$

$$k_t = \hat{r}_t(\sigma(t)^2 + \hat{r}_t)^{-1}.$$

Reference [41]. The Gaussian density and the Gaussian distributions are essential here. The filter system has some analogy with the Kalman-Bucy filter system. It differs from this filter system in that the equation for the conditional covariance depends directly on the observations. A serious difficulty is the integration over the jump measures.

<u>EXAMPLE 5.3.4</u>. A model for optical communication:

$$dx_t = \alpha x_t dt + \beta dv_t, \qquad x_0,$$

$$dn_t = (\mu_0 + \mu_1 x_t^2)\,dt + dm_t, \qquad n_0 = 0,$$

$$\lambda_t = (\mu_0 + \mu_1 x_t^2),$$

where v is a standard Brownian motion process, and x_0: $\Omega \to R$ is a Gaussian distributed random variable. The solution to the stochastic filtering problem for this model is only known locally, between jump times,

$$E[\exp(iu\lambda_t) \mid F_t^n] I_{(\tau_k \le t < \tau_{k+1})}$$
$$= \exp(iu\mu_0) \sum_{j=0}^{k} a_{kj}(t)[1 - iug(t)]^{-j-\frac{1}{2}} / [\sum_{j=0}^{k} a_{kj}(t)],$$

where $(a_{kj}(t), F_t^n, t \in T)$ are adapted stochastic processes for which equations are known, and g: $T \to R$ is a determinstic function. Reference [4]. The characteristic function above is locally a convex combination of characteristic functions of gamma type. The resulting filter system does not seem to be finite dimensional. Aprpoximations to the solution may be attempted.

6. THE MEASURE TRANSFORMATION METHOD

6.1. Introduction

A second method to resolve the stochastic filtering problem is the measure transformation method. To introduce this method we first consider an elementary example.

EXAMPLE 6.1.1. Consider the model with random variables n: $\Omega \to R_+$, λ: $\Omega \to R_+$ such that

$$E[\exp(iun) \mid F^\lambda] = \exp(\lambda(e^{iu}-1)),$$

λ has a Gamma distribution, with density function $p(v) = \beta^{-r} v^{r+1} e^{-v/\beta}/\Gamma(r)$, with $\beta, r \in (0,\infty)$. Thus conditioned on λ, n has a Poisson distribution. The problem is to determine

$$E[\exp(iu\lambda) \mid F^n].$$

We sketch the method. Let $\rho = \lambda^n \exp(-\lambda+1)$. Define a new measure P_0 on (Ω,F) by $P_0(A) = E[I_A \rho^{-1}]$. Then it can be shown that: 1. P_0 is absolute continuous with respect to the original measure P; 2. $E_0[\exp(iun)] = \exp(e^{iu}-1)$; 3. $(F^n,F^\lambda) \in I(P_0)$, or F^n, F^λ are independent under P_0; 4. $P_0 = P_1$ on F^λ; 5.

$$E[\exp(iu\lambda) \mid F^n] = E_0[\exp(iu\lambda)\rho \mid F^n]/E_0[\rho \mid F^n]$$

$$= (1 - iu\beta/(\beta+1))^{-(n+r)}.$$

The above procedure may be considered as a measure theoretic formulation of the Bayes method. The calculation in point five above is straightforward because under P_0, F^n and F^λ are independent.

We can now formulate the measure transformation method to resolve the stochastic filtering problem. It consists of the steps:

1. to perform a measure tranformation such that under the new measure the state process and the observed process are independent;
2. to obtain a semimartingale representation for the unnormalized conditional characteristic function with respect to the new measure. This representation will be in the form of a partial stochastic differential equation.

The advantage of this method is that the calculations under the new measure are easier than under the old measure. Below we briefly sketch the measure transformation method for the case of counting process observations.

6.2. The Counting Process Case

THEOREM 6.2.1. *Given the processes, with respect to a measure* P_1,

$$x: \Omega \times T \to R,$$

$$dn_t = \lambda(x_t)dt + dm_t, \qquad n_0 = 0,$$

where n *is a counting process.*

(a) *Then there exists a probability measure* P_0 *on* (Ω,F) *such that:*

1. $P_1 << P_0$ *with* $\rho_t := E_0[dP_1/dP_0 \mid F_t^n \vee F_\infty^x] =$
$= \exp(\int_0^t \ell n(\lambda(x_s))dn_s - \int_0^t [\lambda(x_s) - 1]ds)$;
2. *under* P_0, n *is a standard Poisson process;*
3. $(F_\infty^n, F_\infty^x) \in I(P_0)$;
4. $P_1 = P_0$ *on* F_∞^x;
5. $E_1[\exp(iux_t) \mid F_t^n] = E_0[\exp(iux_t)\rho_t \mid F_t^n]/E_0[\rho_t \mid F_t^n]$ (*)

(b) *If in addition* x *has the representation*

$$dx_t = f(x_t)dt + dm_{1t}, \qquad x_0,$$

$$d\langle m_1, m_1\rangle_t = \phi_t dt,$$

with $(m_{1t}, F_t, t \in T) \in M_1^c$ *and suitable conditions on* x_0, f, ϕ, *then we have the partial stochastic differential equation*

$$dE_0[\exp(iux_t)\rho_t \mid F_t^n]$$

$$= E_0[iu\rho_t f(x_t)\exp(iux_t) - \tfrac{1}{2} u^2\rho_t\phi_t \exp(iux_t) \mid F_t^n]dt$$

$$+ E_0[\rho_t[\lambda(x_t) - 1]\exp(iux_t) \mid F_{t-}^n](dn_t - dt),\; E_0[\exp(iux_0)].$$

References [3, 5]. We call the process $E_0[\exp(iux_t)\rho_t \mid F_t^n]$ the unnormalized conditional characteristic function. Note that if it is known, then by setting u = 0 one obtains the denumerator in (*), and thus the desired expression via again (*)

7. OPEN QUESTIONS

We mention some open questions for the stochastic filtering problem in the case of jump process observations.

1. Which stochastic dynamical systens lead to stochastic filter systems which are, in some sense, finite dimensional? One difficulty is that the concept of finite dimensionality is not yet clear. However the examples of 5.3 indicate what is understood by this term. The importance of the finite dimensionality of a filter system is clear from the viewpoint of applications. One would hope to obtain sufficient conditions for a stochastic dynamical system such that the associated filter system is finite dimensional. It is possible that differential geometric concepts play a role in answering this question.
2. Can stochastic realization be useful in resolving the stochastic filtering problem? For second order processes stochastic realization theory has provided new insights for the stochastic filtering problem. For jump processes this approach is still undeveloped.
3. How useful is the approximation of a rate process by a finite state Markov process? From a viewpoint of applications the most useful result is the filtering algorithm 5.3.1, for the case where the state process is a finite state Markov process. For the application of this result to concrete problems there are several questions of modelling and implementation.

8. REFERENCES

[1] BARAS, J.S., W.S. LEVINE & T.L. LIN, *Discrete-time point processes in urban traffic queue estimation,* IEEE Trans. Automatic Control, 24 (1979), pp. 12-27.

[2] BARAS, J.S., A.J. DORSEY & W.S. LEVINE, *Estimation of Traffic Platoon Structure from Headway Statistics,* IEEE Trans. Automatic Control, 24 (1979), pp. 553-559.

[3] BOEL, R., P. VARAIYA & E. WONG, *Martingales on Jump Processes; I: Representation Results; II: Applications,* SIAM J. Control, 13 (1975), pp. 999-1061.

[4] BOEL, R.K. & V.E. BENES, *Recursive nonlinear estimation of a diffusion acting as the rate of an observed Poisson process,* IEEE Trans. Information Theory, to appear.

[5] BRÉMAUD, P., *A Martingale Approach to Point Processes,* Ph.D. thesis, Univ. of California, Berkeley, Ca., August, 1972.

[6] BRÉMAUD, P., *The matingale theory of point processes over the real half line admitting the intensity,* Proc. IRIA Coll. Control Theory, June, 1974, Lecture Notes Econ. Math. Systems 107, Springer, Berlin, pp. 519-542.

[7] BRÉMAUD, P., *Estimation de l'état d'une file d'attente et du temps de panne d'une machine par la méthode des semi-martingales,* Adv. in Appl. Prob., 7 (1975), pp. 845-863.

[8] BRÉMAUD, P., *La Méthode des Semi-Martingales en Filtrage quand l'observation est un Processu Ponctuel marque,* in *"Séminaire Probabilités X",* Lecture Notes in Math., volume 511, Springer-Verlag, Berlin, 1975, pp. 1-18.

[9] BRÉMAUD, P., *On the output theorem of queueing, via filtering,* J. Appl. Prob., 15 (1978), pp. 397-405.

[10] BRÉMAUD, P. & J. JACOD, *Processus Punctuels et Martingales: Revue des Résultats récents sur la modélisation et le filtrage,* Adv. in Appl. Prob., 9 (1977), pp. 362-416.

[11] DAVIS, M.H.A., *The Structure of Jump Processes and Related Control Problems,* in *"Stochastic Systems: Modelling, Identification and Optimization II",* Math. Programming Studies, no. 6, North-Holland Publ. Co., Amsterdam, The Netherlands, 1976, pp. 2-14.

[12] DAVIS, M.H.A., *A note on the Poisson disorder problem,* Banach Center Publications, 1 (19), pp. 65-72.

[13] DELLACHERIE, C. & P.A. MEYER, *Probabilités et Potentiel,* Hermann, Paris, 1975.

[14] FISHMAN, P.M. & D.L. SNYDER, *The statistical analysis of space-time point processes,* IEEE Trans. Information Theory, 22 (1976), pp. 257-274.

[15] FROST, P.A., *Examples of linear solutions to nonlinear estimation problems,* Proc. 5th annual Princeton Conf. on Information sciences and systems, 1971, pp. 20-24.

[16] GRANDELL, J., *Doubly stochastic Poisson processes,* Lecture Notes in Math., 529, Springer-Verlag, Berlin, 1976.

[17] HADJIEV, D.I., *On the filtering of semimartingales in case of observations of point processes,* Th. Probab. Appl., 24 (1979), pp. 169-178.

[18] JACOD, J., *Calcul Stochastique et problèmes de martingales,* Lecture Notes in Math., volume 714, Springer-Verlag, Berlin, 1979.

[19] KARP, S., E.L. O'NEILL & R.M. GAGLIARDI, *Communication Theory for the free-space Optical Channel,* Proc. IEEE, 58 (1970), pp. 1611-1626.

[20] KARP, S. & J.R. CLARK, *Photon counting: a problem in classical noise theory,* IEEE Trans. Info-th 16, 6, (Nov. 1970), pp. 672-680.

[21] LIPSTER, R.S. & A.N. SHIRYAYEV, *Statistics of Random Processes, I General Theory; II Applications,* Springer-Verlag, Berlin, 1977.

[22] MACCHI, O. & B.C. PICINBONO, *Estimation and Detection of Weak Optical Signals,* IEEE Trans. Information Theory, 18 (1972), pp. 562-573.

[23] MEYER, P.A., *Un cours sur les intégrales stochastiques,* in *"Séminaire de Probabilités X",* Lecture Notes in Math., volume 511, Springer-Verlag, Berlin, 1975, pp. 245-400.

[24] NEVEU, J., *Processus Ponctuels,* in *"Ecole d'été de Probabilités de Saint-Flour VI - 1976",* Lecture Notes in Mathematics, volume 598, Springer-Verlag, Berlin, 1977, pp. 250-447.

[25] RUDEMO, M., *State estimation for partially observed Markov chains,* J. Math. Anal. Appl., 44 (1973), pp. 581-611.

[26] RUDEMO, M., *Prediction and Smoothing for Partially Observed Markov Chains,* J. Math. Anal. Appl., 49 (1975), pp. 1-23.

[27] SEGALL, A., *A martingale approach to modelling, estimation and detection of jump processes,* Ph.D. thesis, Stanford Univ., Stanford, Cal., Aug. 1973.

[28] SEGALL, A., *Dynamic File Assignment in a Computer Network,* IEEE Trans. Automatic Control, 21 (1976), pp. 161-173.

[29] SEGALL, A. & T. KAILATH, *The modelling of randomly modulated jump processes,* IEEE Trans. Information Theory, 21 (1975), pp. 135-143.

[30] SEGALL, A., M.H.A. DAVIS & T. KAILATH, *Nonlinear filtering with counting observations,* IEEE Trans. Information Theory, 21 (1975), pp. 143-149.

[31] SNYDER, D.L., *Filtering and detection for doubly stochastic Poisson processes,* IEEE Trans. Information Theory, 18 (1972), pp. 91-102.

[32] SNYDER, D.L., *Smoothing for doubly stochastic Poisson processes,* IEEE-Trans. Information Theory, 18 (1972), pp. 558-562.

[33] SNYDER, D.L., *Statistical analysis of dynamic tracer data,* IEEE Trans. Biomedical Engineering, 20 (1973), pp. 11-20.

[34] SNYDER, D.L. & P.M. FISHMAN, *How to track a swarm of fireflies by observing their flashes,* IEEE-T-IT 21, 6 (1975), pp. 692-695.

[35] SNYDER, D.L., *Random point processes,* Wiley, N.Y., 1975.

[36] MAARSEVEEN, M.F.A.M. VAN, *Estimation of counting processes and application to traffic flow - A martingale approach,* Report, Onderafd. Toegepaste Wiskunde, Technische Hogeschool Twente, Enschede, The Netherlands, Sept. 1976.

[37] PUTTEN, C. VAN & J.H. VAN SCHUPPEN, *On stochastic dynamical systems,* Proc. Fourth Int. Symposium on Mathematical Theory of Networks and Systems, July 1979, Delft, The Netherlands, pp. 350-355.

[38] SCHUPPEN, J.H. VAN, *Estimation theory for continuous-time processes, a martingale approach,* Ph.D. thesis, University of California, Berkeley, Ca., Sept. 1973.

[39] SCHUPPEN, J.H. VAN, *Filtering, Prediction and Smoothing for Counting Process Observations, a Martingale Approach,* SIAM J. Appl. Math., 32 (1977), pp. 552-570.

[40] WAN, C.B. & M.H.A. DAVIS, *The General Point Process Disorder Problem,* IEEE Trans. Information Theory, 23 (1977), pp. 538-540.

[41] SNYDER, D.L. & P.M. FISHMAN, *How to track a swarm of fireflies by observing their flashes,* IEEE Trans. Information Theory, 21, (1975), pp. 692-695.